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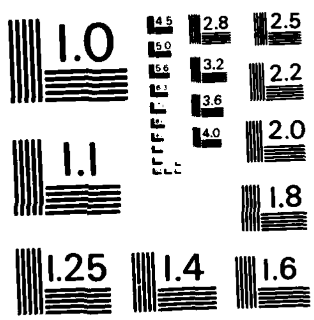

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The University of Connecticut
SCHOOL OF ENGINEERING
Storrs, Connecticut 06268

Robust Prediction and Interpolation For Vector Stationary Processes

by

Haralampos Tsaknakis
Dimitri Kazakos
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Technical Report TR-82-7

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Department of
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Haralampos Tsaknakis, Dimitri Kazakos, and P. Papantoni-Kazakos
University of Connecticut, Storrs, Connecticut 06268
and
University of Virginia, Charlottesville, Virginia 22901

Abstract

Asymptotic linear prediction and interpolation, for statistically contaminated vector stationary processes is considered. Both prediction and interpolation are then formalized as stochastic games with saddle point solutions. The existence of unique solutions on convex and closed classes of vector stationary processes is shown. Then, those solutions are found and analyzed, for two specific classes of vector stationary processes.

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1. Introduction

The prediction and interpolation problems for stationary processes have received considerable attention for a number of years. The bulk of the work concentrates around scalar stationary processes and the parametric model. The assumption there is that the measure of the stationary process is well known. The initial significant results on prediction and interpolation for the parametric model were given by Kolmogorov [2] and Wiener [1]. Most of the to date results on the same model can be found in Hannan [12]. There, both scalar and vector stationary processes are considered.

Strictly speaking, the term prediction refers to the extraction of a datum from the process, when a number of past process data have been observed noiselessly. The term interpolation refers to the same extraction, when past as well as future noiseless process data are available. The two terms are extended sometimes to include noisy observation data. Some results on those extended problems, and for the parametric model, can be found in [4] and [6]. We point out here that the majority of studies on the extended problems consider asymptotic and linear prediction and interpolation operations.

The last few years considerable attention has been given to the robust extended prediction problem. Little attention has been given to the robust nonextended interpolation problem. The robust model is nonparametric, and the assumption is that the measure that generates the stationary process is not well known. The existing work on robust extended prediction and interpolation concentrates around scalar stationary processes, linear asymptotic prediction and interpolation operations, and noisy observation data. Representative results here include robust Wiener and Kalman filtering for scalar stationary processes, and can be found in [3], [5], [7], [10], and [11]. Related work on time series outliers can be found in [8]. The work in [9] considers the robust nonextended prediction problem, for linearly contaminated scalar stationary processes. The robust solution is found there in the class of asymptotic linear prediction operations. In [13], a game theoretic formulation of the robust extended prediction problem is presented. There, statistical contamination on the measure of the processes is assumed, and it is concluded that linear prediction operations may not be robust unless preceeded by appropriate nonlinear transformations. In [14], some performance bounds on the robust extended prediction and interpolation problems are given.

In the present paper, we consider the robust nonextended prediction and interpolation problems for vector stationary processes. Vector processes have not been treated in the robust literature, and they present some interesting peculiarities. In particular, we consider prediction and interpolation in the absence of noise. We assume that the vector process is a member of a convex and closed class of stationary vector processes. We adopt asymptotic and linear prediction and interpolation operations. To maintain consistency with the results in [13], we assume that a nonlinear stationary operation has preceeded the linear prediction and interpolation operations. As exhibited in [13], an appropriate nonlinear stationary operation maps the convex and closed class of vector stationary processes onto another convex and closed class of such processes. We formulate the prediction and interpolation problems as stochastic games with saddle point solutions. We find these solutions explicitly, for two convex and closed classes of vector stationary processes. One of the classes represents linear contamination of a nominal vector process. The other class includes vector processes with fixed energy on prespecified frequency quantiles.

The organization of the paper is as follows. In section 2, we formulate the prediction and interpolation games and we prove the existence of unique solutions. In section 3, we analyze the above games for the class representing linear contamination of a nominal vector process. In section 4, we analyze the prediction and interpolation games for the class of vector processes with fixed energy on prespecified frequency quantiles. In section 5, we present some conclusions.

2. Preliminaries

We consider the prediction and interpolation problems for stationary vector processes. We assume that the statistical structure of the vector process is incompletely known. We model this incomplete knowledge by a convex and closed family $\mathcal{F}$ of stationary vector processes. In the prediction problem, we adopt the class S_p of one-step, asymptotic linear predictors. In the interpolation problem, we adopt the class S_i of linear interpolators that operate on the infinite past and the infinite future data sequences from the vector stationary process. We select the mean square error as the payoff function and we formalize a saddle point game on $\mathcal{F}xS_p$ and $\mathcal{F}xS_i$ respectively. We call the corresponding solutions of the game, robust. We should point out here that our solutions may not satisfy the properties of qualitative robustness. Indeed, as exhibited in [13], linear operations on a stationary process may not satisfy the sufficient conditions for qualitative robustness. These sufficient conditions will be satisfied, however, if an appropriate nonlinear operation preceeds the linear predictor or interpolator [13]. Thus, in this paper, we will assume that the class $\mathcal{F}$ of vector stationary processes is, in general, induced by an appropriate stationary nonlinear operation on the original class $\mathcal{F}_o$ of such processes. We will first formalize the games on $\mathcal{F}xS_p$ and $\mathcal{F}xS_i$ for an arbitrary convex and closed set $\mathcal{F}$. Then, we will consider some specific $\mathcal{F}$ choices.

Let $\mathcal{F}$ be some convex and closed family of discrete-time, n-dimensional, complex-valued, vector stationary processes. Let μ denote the measure of some process in $\mathcal{F}$, and let $\underline{F}_\mu(\omega)$ denote the spectral distribution matrix of the process μ in $\mathcal{F}$. The spectral distribution matrix $\underline{F}_\mu(\omega)$ is defined on $[-\pi,\pi]$, it has Hermitian nonnegative increments $\underline{F}_\mu(\omega_1)-\underline{F}_\mu(\omega_2)$; $\omega_1 \geq \omega_2$, it is continuous from the right, and it satisfies the equation $\underline{F}_\mu(-\pi) = \underline{0}$. Furthermore, $\underline{F}_\mu(\omega)$ defines a finite matrix measure on the Borel field B_π of the measurable space $[-\pi,\pi]$, in the sense that $||\underline{F}_{\mu}([-\pi,\pi])|| = ||\underline{F}_{\mu}(\pi)|| < \infty$, under any matrix norm $||\cdot||$. Throughout this paper, we will assume that the above finite matrix measures are absolutely continuous with respect to the Lebesque measure on $[-\pi,\pi]$, for all μ in F. Then, for any μ in F, the spectral density matrix $\underline{f}_{\mu}(\omega) \triangleq \frac{d}{d\omega} \underline{F}_{\mu}(\omega)$ exists, and it is, in general, a Hermitian nonnegative definite matrix whose elements are Lebesque integrable functions on the measurable space $([-\pi,\pi], B_{\pi})$.

Let S_{μ} be the space of all nxn complex matrices $A(\omega)$ with the property:

$$A(\omega) \in S_{\mu} \leftrightarrow \operatorname{tr} \int_{-\pi}^{\pi} A^{*T}(\omega)\ \underline{f}_{\mu}(\omega)\ A(\omega)\ d\omega < \infty \tag{1}$$

; where μ some stationary process in F, tr means trace, and * stands for conjugate.

Then, S_{μ} is a Hilbert space with inner product and norm given respectively by the following expressions:

$$(A_1(\omega),\ A_2(\omega)) \triangleq \operatorname{tr} \int_{-\pi}^{\pi} A_1^{*T}(\omega)\ \underline{f}_{\mu}(\omega)\ A_2(\omega)\ d\omega\ ;\ A_1(\omega),\ A_2(\omega) \in S_{\mu} \tag{2}$$

$$||A_1(\omega)||_{S_{\mu}} \triangleq (A_1(\omega),\ A_1(\omega))^{1/2} \tag{3}$$

Let S_p and S_i be the sets of all matrix polynomials of the form $\sum_{k=1}^{\infty} A_k\ e^{jkw}$ and $\sum_{k=-\infty}^{\infty} A_k\ e^{jkw}$ respectively; where $\{A_k\}$ is any square summable sequence of constant nxn matrices. It is easily seen that $S_p \subset S_{\mu}$ and $S_i \subset S_{\mu}$; $\forall\ \mu \in F$. We will denote by $\underline{g}_p$ members of the set S_p. We will denote by $\underline{g}_i$ members of the set S_i.

Let I_n be the identity n-dimensional matrix. Then, $I_n \in S_{\mu}$; $\forall\ \mu \in F$. Given $\mu \in F$, it is well-known from the classical theory of mean square linear prediction and interpolation [12], that in the frequency domain both problems correspond to appropriate projections of the identity matrix I_n. In particular, in the prediction problem the projection of I_n onto the subspace $S_p \subset S_\mu$ is sought. In the interpolation problem, the projection of I_n onto the subspace $S_i \subset S_\mu$ is sought, instead. For given μ in F, the above projections are realized by two infima $\inf_{\underline{g}_p \in S_p} e_p(\mu,\underline{g}_p)$ and $\inf_{\underline{g}_i \in S_i} e_i(\mu,\underline{g}_i)$ respectively, where:

$$e_p(\mu,\underline{g}_p) \triangleq \mathrm{tr}\int_{-\pi}^{\pi} (I_n-\underline{g}_p(\omega))^{*T}\, \underline{f}_\mu(\omega)\, (I_n-\underline{g}_p(\omega))\, d\omega \;;\; \underline{g}_p \in S_p \tag{4}$$

$$e_i(\mu,\underline{g}_i) = \mathrm{tr}\int_{-\pi}^{\pi} (I_n-\underline{g}_i(\omega))^{*T}\, \underline{f}_\mu(\omega)\, (I_n-\underline{g}_i(\omega))\, d\omega \;;\; \underline{g}_i \in S_i \tag{5}$$

It is also well-known [15] that:

$$e_p(\mu) \triangleq \inf_{\underline{g}_p \in S_p} e_p(\mu,\underline{g}_p) = \min_{\underline{g}_p \in S_p} e_p(\mu,\underline{g}_p) = \exp\left\{(2\pi n)^{-1}\int_{-\pi}^{\pi} \mathrm{tr}[\ell og\; 2\pi\, \underline{f}_\mu(\omega)]\, d\omega\right\};\; \mu \in F \tag{6}$$

$$e_i(\mu) \triangleq \inf_{\underline{g}_i \in S_i} e_i(\mu,\underline{g}_i) = \min_{\underline{g}_i \in S_i} e_i(\mu,\underline{g}_i) = 4\pi^2\, \mathrm{tr}\left[\int_{-\pi}^{\pi} \underline{f}_\mu^{-1}(\omega)\, d\omega\right]^{-1} \;;\; \mu \in F \tag{7}$$

We now consider games on $F\mathrm{x}S_p$ and $F\mathrm{x}S_i$, with payoff functions given respectively by $e_p(\mu,\underline{g}_p)$ in (4), and $e_i(\mu,\underline{g}_i)$ in (5). We are seeking pairs $(\mu_p^*,\underline{g}_p^*)$ on $F\mathrm{x}S_p$ and $(\mu_i^*,\underline{g}_i^*)$ on $F\mathrm{x}S_i$, such that:

$$\forall\, \mu \in F \;;\; e_p(\mu,\underline{g}_p^*) \le e_p(\mu_p^*,\underline{g}_p^*) \le e_p(\mu_p^*,\underline{g}_p) \;;\; \forall\, \underline{g}_p \in S_p \tag{8}$$

$$\forall\, \mu \in F \;;\; e_i(\mu,\underline{g}_i^*) \le e_i(\mu_i^*,\underline{g}_i^*) \le e_i(\mu_i^*,\underline{g}_i) \;;\; \forall\, \underline{g}_i \in S_i \tag{9}$$

If the pairs in (8) and (9) exist, we call them solutions of the corresponding games. We proceed with a theorem whose proof is in the appendix.

Theorem 1

Let F be a convex and closed family of n-dimensional vector stationary processes with spectral distribution matrices satisfying the properties stated in this section. Let $e_p(\mu)$ and $e_i(\mu)$ be given by (6) and (7) respectively. Then, there exist unique measures μ_p^* and μ_i^* in F, such that:

$$e_p(\mu_p^*) = \sup_{\mu \varepsilon F} e_p(\mu) \tag{10}$$

$$e_i(\mu_i^*) = \sup_{\mu \varepsilon F} e_i(\mu) \tag{11}$$

Let $\underline{g}_p^* \varepsilon S_p$ and $\underline{g}_i^* \varepsilon S_i$ be such that:

$$e_p(\mu_p^*,\underline{g}_p^*) = e_p(\mu_p^*)$$

$$e_i(\mu_i^*,\underline{g}_i^*) = e_i(\mu_i^*)$$

Then, the pairs $(\mu_p^*,\underline{g}_p^*)$ and $(\mu_i^*,\underline{g}_i^*)$ are the unique solutions of the games in (8) and (9) respectively.

Theorem 1 and its proof are parallel to theorem 1 in [14]. In the latter, more general asymptotic as well as nonasymptotic filters and smoothers are included. According to the above theorem, to find solutions for the games in (8) and (9), it is sufficient to search for measures that satisfy the suprema in (10) and (11).

To this point, we considered arbitrary convex and closed families F of stationary vector processes. To find explicit solutions for the games in theorem 1, we will adopt two specific such families. Since in the error expressions in (4) and (5), the measure of the vector process appears through its spectral density matrix only, we will define the two families through their spectral characteristics.

The first family we consider satisfies the general properties stated in the beginning of this section, and it corresponds to linear contamination of a nominal measure μ_0. In particular, denoting by $\underline{f}_0(\omega)$ the well-known spectral density matrix of the measure μ_0, we define this family in the frequency domain as follows:

(A) $F_{L,\varepsilon} = \Big\{ \underline{f}(\omega) : \underline{f}(\omega) = (1-\varepsilon)\, \underline{f}_0(\omega) + \varepsilon\, \underline{h}(\omega) \;;\; \omega \in [-\pi,\pi]$

; where ε given and such that: $0 < \varepsilon < 1$,

$\underline{f}_0(\omega)$ well-known positive definite Hermitian matrix,

$\underline{h}(\omega)$ nonnegative definite Hermitian matrix satisfying the energy

constraint: $(2\pi)^{-1}\, \mathrm{tr} \int_{-\pi}^{\pi} \underline{h}(\omega)\, d\omega \le W$, for given positive W. $\Big\}$

The second family we consider satisfies again the general properties in the beginning of this section, and it corresponds to fixed energy within a finite number of subsets of the Borel field B_π. Specifically, this family is defined as follows:

(B) $F_Q = \Big\{ \underline{f}(\omega) : \mathrm{tr} \int_{A_i} \underline{f}(\omega)\, d\omega = c_i \;;\; i=1,2,\ldots,k,\; \mathrm{tr} \int_{-\pi}^{\pi} \underline{f}(\omega)\, d\omega = c$

; where $A_i \in B_\pi$; $\forall i$, $A_i \cap A_j = \phi$; $\forall\, i \neq j$, $\bigcup_{i=1}^{k} A_i \subset [-\pi,\pi]$,

$c > \sum_{i=1}^{k} c_i$, and $\underline{f}(\omega)$ positive definite Hermitian matrix $\Big\}$

Given two nxn spectral density matrices $\underline{f}_1(\omega)$ and $\underline{f}_2(\omega)$, define the metric

$\sum_{i=1}^{n} |\lambda_i^{(1)}(\omega) - \lambda_i^{(2)}(\omega)|$; where $\{\lambda_i^{(j)}(\omega);\ 1 \le i \le n\}$ the eigenvalues of the matrix

$\underline{f}_j(\omega)$; j=1,2. Then, both families $F_{L,\varepsilon}$ and F_Q are clearly convex and closed with respect to that metric; thus they satisfy the conditions in theorem 1. Therefore, it is sufficient to search for the suprema $\sup_{\mu \in F_{L,\varepsilon}} e_p(\mu)$, $\sup_{\mu \in F_Q} e_p(\mu)$, $\sup_{\mu \in F_{L,\varepsilon}} e_i(\mu)$,

$\sup_{\mu \in F_Q} e_i(\mu)$.

In section 3, we will solve the prediction and interp.·ation games for the class $\mathcal{F}_{L,\varepsilon}$. In section 4, we will solve the same games for the class $\mathcal{F}_Q$.

3. The Solution of the Games for $\mathcal{F}_{L,\varepsilon}$

Due to the results in theorem 1, the solution of the prediction and the interpolation games corresponds respectively to the measures that realize the suprema of the errors $e_p(\mu)$ and $e_i(\mu)$ in (6) and (7). In this section, we are searching for the above suprema on $\mathcal{F}_{L,\varepsilon}$; where the class $\mathcal{F}_{L,\varepsilon}$ is given by (A) in section 2.

Given some spectral density matrix $\underline{f}(\omega)$ in $\mathcal{F}_{L,\varepsilon}$ let us denote by $\{\lambda_j(\omega)$; $j=1,\ldots,n\}$ its eigenvalues. Since, $\underline{f}(\omega)$ is positive definite, $\lambda_j(\omega) \geq 0$; $\forall$ j, $\forall\ \omega\varepsilon[-\pi,\pi]$. Let us denote by $\{\lambda_j^o(\omega)$; $j=1,\ldots,n\}$ the eigenvalues of the nominal spectral density matrix $\underline{f}_0(\omega)$. Let us define two scalar functions on $[-\pi,\pi]$, that we will use later in our derivations. In particular, if $\{\lambda_j(\omega)$; $1 \leq j \leq n\}$ are the eigenvalues of the spectral density matrix $\underline{f}(\omega)$ in $\mathcal{F}_{L,\varepsilon}$, we define:

$$\lambda_{min}(\omega) \triangleq \min_{1\leq j\leq n} \lambda_j(\omega) \quad ; \quad \omega \ \varepsilon\ [-\pi,\pi] \tag{12}$$

$$\lambda^o_{max}(\omega) \triangleq \max_{1\leq j\leq n} \lambda^o_j(\omega) \quad ; \quad \omega \ \varepsilon\ [-\pi,\pi] \tag{13}$$

At this point we observe that for arbitrary matrix A with eigenvalues $\{\lambda_j\}$, we have the known identities: $\mathrm{tr}\{\ell og\ A\} = \ell og\{\det A\}$ and $\det A = \prod_j \lambda_j$; where det denotes determinant. If we apply the above identities to the error expression $e_p(\mu)$ in (6), we obtain:

$$e_p(\mu) \triangleq e_p(\underline{f}) = 2\pi \exp\left\{ (2\pi n)^{-1} \int_{-\pi}^{\pi} \ell og\ \left(\prod_{j=1}^{n} \lambda_j(\omega)\right) d\omega \right\} \tag{14}$$

; where $\{\lambda_j(\omega)$; $1 \leq j \leq n\}$ the eigenvalues of the spectral density matrix $\underline{f}(\omega)$ that corresponds to the measure μ.

For completion, we reexpress the interpolation error $e_i(\mu)$ in (7):

$$e_i(\mu) \triangleq e_i(\underline{f}) = 4\pi^2 \operatorname{tr}\left[\int_{-\pi}^{\pi} \underline{f}^{-1}(\omega)\, d\omega\right]^{-1} \tag{15}$$

From expression (14), we observe that the prediction error $e_p(\underline{f})$ is solely a function of the eigenvalues of the matrix $\underline{f}(\omega)$. The eigenvectors of $\underline{f}(\omega)$ do not appear explicitly in the error expression $e_p(\underline{f})$. At the same time, there are no explicit statements in the description of the class $F_{L,\varepsilon}$ in (A), referring to the eigenvectors of the spectral density matrices $\underline{f}(\omega)$. For those reasons, we will take two extreme directions. We will first assume that the class $F_{L,\varepsilon}$ includes spectral densities with no restrictions on the eigenvectors, besides the ones implied by the semipositive definite requirement on the matrices $\underline{h}(\omega)$. Then, we will consider a subclass $F'_{L,\varepsilon}$ contained in $F_{L,\varepsilon}$. The subclass $F'_{L,\varepsilon}$ will contain only those matrices $\underline{f}(\omega)$ in $F_{L,\varepsilon}$ that have eigenvectors identical with those of the nominal spectral density matrix $\underline{f}_o(\omega)$. That is, the spectral density matrices in $F'_{L,\varepsilon}$ will have identical projections on the directions determined by the eigenvectors of the matrix $\underline{f}_o(\omega)$. We observe from expression (15) that the interpolation error $e_i(\underline{f})$ is, in general, a function of both the eigenvalues and the eigenvectors of the spectral density matrix $\underline{f}(\omega)$. Thus, we will find a suboptimal solution of the interpolation game on $F_{L,\varepsilon}$. We will solve the prediction game for both the classes $F_{L,\varepsilon}$ and $F'_{L,\varepsilon}$. We first state formally the description of the class $F'_{L,\varepsilon}$.

(C) $F'_{L,\varepsilon} = \left\{ \underline{f}(\omega) : \underline{f}(\omega) \in F_{L,\varepsilon}, \text{ and } \underline{f}(\omega) \text{ has eigenvectors identical with those of the matrix } \underline{f}_0(\omega). \right\}$

We will first find the solution of the prediction game (or equivalently the supremum of $e_p(\underline{f})$) within the class $F_{L,\varepsilon}$. To do that we will use a trick. We will solve the corresponding optimization problem within some subclass F of $F_{L,\varepsilon}$. Then, we will show that the so obtained solution is sufficient, in the sense that it also solves the optimization problem in $F_{L,\varepsilon}$. We will first define the subclass F and we will prove some of its properties.

(D) $F = \left\{ \begin{array}{l} \underline{f}(\omega) : \underline{f}(\omega) \in F_{L,\varepsilon}, \text{ and } \lambda_{min}(\omega) \geq (1-\varepsilon)\ \lambda^{o}_{max}(\omega)\ ;\ \forall\ \omega \in [-\pi,\pi]; \\ \qquad \text{where } \lambda_{min}(\omega) \text{ and } \lambda^{o}_{max}(\omega) \text{ are given by expressions (12)} \\ \qquad \text{and (13)} \end{array} \right\}$

Proposition 1

The class F is nonempty, convex and closed, and it is contained in $F_{L,\varepsilon}$.

Proof

i) Let $\underline{f}(\omega)$ be some spectral density matrix in $F_{L,\varepsilon}$. Then, due to the nonnegative definite requirement on $\underline{h}(\omega)$, we have that the matrix $\underline{f}(\omega) - (1-\varepsilon)\underline{f}_0(\omega)$ is nonnegative definite. Let $X(\omega)$ be the eigenvector of $\underline{f}_0(\omega)$ that achieves $\lambda^{o}_{max}(\omega)$. Due to the complete freedom of the eigenvectors in $F_{L,\varepsilon}$, there exists some $\underline{f}(\omega)$ in it, such that $X^T(\omega)\underline{f}(\omega)X(\omega) = \lambda_{min}(\omega)$. Thus, there exists some $\underline{f}(\omega)$ in $F_{L,\varepsilon}$ such that:

$$X^T(\omega)\underline{f}(\omega)X(\omega) - (1-\varepsilon)X^T(\omega)\underline{f}_0(\omega)X(\omega) = \lambda_{min}(\omega) - (1-\varepsilon)\ \lambda^{o}_{max}(\omega) \geq 0$$

and nonemptiness is established.

ii) Let $\underline{f}^{(1)}(\omega)$ and $\underline{f}^{(2)}(\omega)$ be in F. Let α be some constant in $(0,1)$. Then, $[\alpha\ \underline{f}^{(1)}(\omega) + (1-\alpha)\ \underline{f}^{(2)}(\omega)] \in F_{L,\varepsilon}$. Let $X(\omega)$ be the eigenvector that achieves the $\lambda_{min,\ \alpha\ \underline{f}^{(1)} + (1-\alpha)\ \underline{f}^{(2)}}(\omega)$ for the matrix $\alpha\ \underline{f}^{(1)}(\omega) + (1-\alpha)\underline{f}^{(2)}(\omega)$. Then,

$$\lambda_{min,\alpha\underline{f}^{(1)}+(1-\alpha)\underline{f}^{(2)}}(\omega) = X^T(\omega)[\alpha\underline{f}^{(1)}(\omega) + (1-\alpha)\underline{f}^{(2)}(\omega)]\ X(\omega) = \alpha\ X^T(\omega)\ \underline{f}^{(1)}(\omega)\ X(\omega) +$$

$$+ (1-\alpha)\ X^T(\omega)\ \underline{f}^{(2)}(\omega)\ X(\omega) \geq \alpha\ \lambda_{min,\underline{f}^{(1)}}(\omega) + (1-\alpha)\ \lambda_{min,\underline{f}^{(2)}}(\omega) \geq (1-\varepsilon)\ \lambda^{o}_{max}(\omega)$$

Thus, convexity has been proven. Closeness is straightforward due to the inclusion of equality in $\lambda_{min}(\omega) \geq (1-\varepsilon)\ \lambda^{o}_{max}(\omega)$.

iii) Let $\underline{f}(\omega) \in F$. Let $X(\omega)$ any vector. Then,

$$\frac{X^T(\omega)\ \underline{f}(\omega)\ X(\omega)}{X^T(\omega)\ X(\omega)} \geq \lambda_{min}(\omega) \geq (1-\varepsilon)\ \lambda^o_{max}(\omega) \geq (1-\varepsilon)\ \frac{X^T(\omega)\ \underline{f}_0(\omega)\ X(\omega)}{X^T(\omega)\ X(\omega)}$$

Or

$$\frac{X^T(\omega)\ \underline{f}(\omega)\ X(\omega)}{X^T(\omega)\ X(\omega)} - (1-\varepsilon)\ \frac{X^T(\omega)\ \underline{f}_0(\omega)\ X(\omega)}{X^T(\omega)\ X(\omega)} \geq 0\ ;\ \forall\ X(\omega)$$

Thus, $\underline{f}(\omega) \in F_{L,\varepsilon}$, and $F \subset F_{L,\varepsilon}$.

Due to proposition 1, the class F in (D) satisfies the properties in theorem 1. Thus, the games in (8) and (9), on FxS_p and FxS_i respectively are equivalent to establishing the suprema of $e_p(\underline{f})$ and $e_i(\underline{f})$ on F. Since F is contained in $F_{L,\varepsilon}$, the energy constraint $(2\pi)^{-1}$ tr $\int_{-\pi}^{\pi} \underline{h}(\omega)\, d\omega \leq W$ is also implied in F. Writing $\varepsilon\ \underline{h}(\omega) = \underline{f}(\omega) - (1-\varepsilon)\ \underline{f}_0(\omega)$, and tr $\int_{-\pi}^{\pi} \underline{f}(\omega)\, d\omega = \int_{-\pi}^{\pi} \left[\sum_{i=1}^{n} \lambda_i(\omega)\right] d\omega$, we obtain the following form of the above energy constraint:

$$\int_{-\pi}^{\pi} \left[\sum_{i=1}^{n} \lambda_i(\omega)\right] d\omega \leq (1-\varepsilon) \int_{-\pi}^{\pi} \left[\sum_{i=1}^{n} \lambda^o_i(\omega)\right] d\omega + \varepsilon\ 2\pi\ W \qquad (16)$$

The maximization of the errors $e_p(\underline{f})$ and $e_i(\underline{f})$ on F reduces then to two optimization problems with two inequality constraints on just the eigenvalues of the matrices $\underline{f}(\omega)$. The optimization problems have unique solutions, due to theorem 1. Using the expressions in (14) and (16), and the description of F in (D), we first state the optimization problem for $e_p(\underline{f})$ on F and its solution in a lemma.

Lemma 1

The supremum of $e_p(\underline{f})$ on F in (D) corresponds to the solution of the following constraint optimization problem:

$$\sup \int_{-\pi}^{\pi} \ell og \left[\prod_{j=1}^{n} \lambda_j(\omega) \right] d\omega$$

subject to:

$$\int_{-\pi}^{\pi} \left[\sum_{i=1}^{n} \lambda_i(\omega) \right] d\omega \leq (1-\varepsilon) \int_{-\pi}^{\pi} \left[\sum_{i=1}^{n} \lambda_i^o(\omega) \right] d\omega + \varepsilon \, 2\pi \, W$$

$$\lambda_j(\omega) \geq (1-\varepsilon) \, \lambda_{max}^o(\omega) \quad ; \; \forall \, j \; , \; \forall \, \omega \in [-\pi,\pi]$$

; where $\lambda_{max}^o(\omega)$ is given by (13).

The unique solution of the above optimization problem is represented by an n-tuple $(\lambda_1^*(\omega),\ldots,\lambda_n^*(\omega))$ of eigenvalues; where:

$$\forall j \; ; \; \lambda_j^*(\omega) = \lambda^*(\omega) = \begin{cases} (1-\varepsilon) \, \lambda_{max}^o(\omega) & ; \; \omega\text{: } (1-\varepsilon) \, \lambda_{max}^o(\omega) > \gamma \\ \gamma & ; \; \omega\text{: } (1-\varepsilon) \, \lambda_{max}^o(\omega) \leq \gamma \end{cases} \tag{17}$$

and γ is a constant uniquely defined.

We observe that the solution in the lemma does not involve eigenvectors at all. Since there are no eigenvector restrictions in the class F, the implication is that the problem is solved by any spectral density matrix whose eigenvalues are given by (17) and whose eigenvectors are unrestricted. Thus, the uniqueness of the solution refers to eigenvalues only. We point out the similarities of the solution in (17) with Hosoya's solution [11], for scalar stationary processes. We exhibit this solution graphically, in figure 1. We now present the proof of the lemma.

Proof of Lemma 1

To find the solution of the optimization problem in lemma 1, we write the Euler-Langrange equations and apply the Kuhn-Tucker conditions:

$$\lambda_i^{-1}(\omega) - \gamma^{-1} - \mu_i(\omega) = 0 \quad ; 1 \le i \le n$$

$$\mu_i(\omega)[\lambda_i(\omega) - (1-\varepsilon)\,\lambda^o_{max}(\omega)] = 0 \quad ; 1 \le i \le n$$

; where $\mu_i(\omega)$; $1 \le i \le n$ the Euler-Langrange multipliers.

From the Euler-Langrange equations we find that the optimal solution satisfies the condition:

$$\lambda_j^*(\omega) = \lambda^*(\omega) \quad ; \quad 1 \le j \le n$$

; where $\lambda^*(\omega)$ is equal to either $(1-\varepsilon)\,\lambda^o_{max}(\omega)$ or to a constant γ such that $\gamma > (1-\varepsilon)\,\lambda^o_{max}(\omega)$. The constant γ, as well as the regions E_γ and E^c_γ on which $\lambda^*(\omega)$ is respectively equal to $(1-\varepsilon)\,\lambda^o_{max}(\omega)$ or to γ are determined by the following equation:

$$\int_{-\pi}^{\pi} \lambda^*(\omega)\, d\omega = (1-\varepsilon)\int_{E_\gamma} \lambda^o_{max}(\omega)\, d\omega + \gamma\int_{E^c_\gamma} d\omega =$$

$$= n^{-1}\left\{(1-\varepsilon)\sum_{i=1}^{n}\int_{-\pi}^{\pi} \lambda^o_i(\omega)\, d\omega + \varepsilon\, 2\pi\, W\right\} \qquad (18)$$

We wish to prove that (18) has a unique solution with respect to γ and E_γ. Let us first define for simplicity in notation:

$$\int_{-\pi}^{\pi} \lambda^o_{max}(\omega)\, d\omega = V \qquad (19)$$

$$n^{-1}\left\{(1-\varepsilon)\sum_{i=1}^{n}\int_{-\pi}^{\pi}\lambda_i^o(\omega)\,d\omega+\varepsilon\,2\pi\,W\right\}=\quad Q \tag{20}$$

Substituting expressions (19) and (20) in expression (18), we obtain:

$$\alpha(\gamma)\triangleq\int_{E_\gamma^c}[\gamma-(1-\varepsilon)\,\lambda_{max}^o(\omega)]\,d\omega=Q-(1-\varepsilon)\,V \tag{21}$$

It is easily shown that $\alpha(\gamma)$ is monotonically nondecreasing with γ, taking values from 0 to ∞. Also, $Q-(1-\varepsilon)\,V>0$. Therefore, the equation in (21) is satisfied for a unique positive γ value. The proof is now complete.

We will now complete the search for the solution of the prediction game in (8) on $\mathcal{F}_{L,\varepsilon}xS_p$, by showing that it is sufficient to place the game on $\mathcal{F}xS_p$ instead; where $\mathcal{F}$ is given by (D). We express this result in a theorem.

Theorem 2

The solution of the prediction game on $\mathcal{F}xS_p$, given by expression (17) in lemma 1, is also the solution of the prediction game on $\mathcal{F}_{L,\varepsilon}xS_p$; where $\mathcal{F}_{L,\varepsilon}$ is described by (A).

Proof

Due to the established uniqueness of the solution on $\mathcal{F}_{L,\varepsilon}xS_p$ (by theorem 1) as well as the solution on $\mathcal{F}xS_p$, it suffices to show that the prediction error $e_p(\underline{f})$, for some $\underline{f}$ in $\mathcal{F}_{L,\varepsilon}-\mathcal{F}$, is bounded from above by the prediction error $e_p(\underline{f}')$, for some $\underline{f}'$ in $\mathcal{F}$. We define $\mathcal{F}_{L,\varepsilon}-\mathcal{F}$ such that $(\mathcal{F}_{L,\varepsilon}-\mathcal{F})\bigcup\mathcal{F}=\mathcal{F}_{L,\varepsilon}$, and $(\mathcal{F}_{L,\varepsilon}-\mathcal{F})\bigcap\mathcal{F}=\phi$.

Let $\underline{f}\in(\mathcal{F}_{L,\varepsilon}-\mathcal{F})$. Then, $\lambda_{min,\underline{f}}(\omega)<(1-\varepsilon)\,\lambda_{max}^o(\omega)$, and there exists at least one eigenvalue of $\underline{f}(\omega)$ such that it is less than $(1-\varepsilon)\,\lambda_{max}^o(\omega)$ for some ω

values. Let the eigenvalues $\lambda_1(\omega),\ldots,\lambda_k(\omega)$; $k \leq n$ have this quality. For the eigenvalue $\lambda_j(\omega)$; $j \leq k$ let us create a new eigenvalue $\lambda_j'(\omega)$ by replacing $\lambda_j(\omega)$ by $(1-\varepsilon)\ \lambda^o_{max}(\omega)$ on those ω values where $\lambda_j(\omega)$ is less than $(1-\varepsilon)\ \lambda^o_{max}(\omega)$. Let us maintain the eigenvalues $\{\lambda_i(\omega)\ ;\ i > k\}$ unchanged. The so created n-tuple $(\lambda_1'(\omega),\ldots,\ \lambda_n'(\omega))$ of eigenvalues clearly satisfies the following inequality (due to the monotonicity of the logorithmic function):

$$\int_{-\pi}^{\pi} \ell og\ [\prod_{i=1}^{n} \lambda_i(\omega)]\ d\omega \leq \int_{-\pi}^{\pi} \ell og\ [\prod_{i=1}^{n} \lambda_i'(\omega)]\ d\omega$$

By construction, any spectral density matrix $\underline{f}'(\omega)$ with eigenvalues the n-tuple $(\lambda_1'(\omega),\ldots,\lambda_n'(\omega))$ is in F. Thus, the proof of the theorem is complete.

We will complete the coverage of the prediction game for this section by finding the solution on $F'_{L,\varepsilon} x S_p$; where the class $F'_{L,\varepsilon}$ is given by (C). We will proceed towards that direction by first stating the properties of the class $F'_{L,\varepsilon}$ in a proposition.

Proposition 2

The class $F'_{L,\varepsilon}$ is nonempty, convex and closed, it is contained in $F_{L,\varepsilon}$, and it is such that:

$$\underline{f}(\omega)\ \varepsilon\ F'_{L,\varepsilon} \rightarrow \lambda_i(\omega) \geq (1-\varepsilon)\ \lambda^o_i(\omega)\ ;\ \forall\ i,\ \forall\ \omega\ \varepsilon\ [-\pi,\pi]$$

Proof

The class is clearly nonempty since it contains at least $\underline{f}_0(\omega)$. It is also trivially convex and closed and contained in $F_{L,\varepsilon}$. Now, since for each $\underline{f}(\omega)$ in $F'_{L,\varepsilon}$, $\underline{f}(\omega)$ and $\underline{f}_0(\omega)$ have the same eigenvectors, the eigenvalues of the difference $\underline{f}(\omega) - (1-\varepsilon)\ \underline{f}_0(\omega)$ are given by $\{\lambda_i(\omega) - (1-\varepsilon)\ \lambda^o_i(\omega)\ ;\ 1 \leq i \leq n\}$. Since

$\underline{f}(\omega) - (1-\varepsilon)\ \underline{f}_0(\omega)$ should be nonnegative definite for every $\underline{f}(\omega)$ in $F'_{L,\varepsilon}$, we conclude then that it is necessary that $\lambda_i(\omega) \geq (1-\varepsilon)\ \lambda_i^o(\omega)$; $\forall$ i, $\forall\ \omega \in [-\pi,\pi]$. The proof of the proposition is now complete.

Due to proposition 2, the class $F'_{L,\varepsilon}$ satisfies the necessary properties in theorem 1. Thus, the prediction game in (8) on $F'_{L,\varepsilon} x S_p$ is equivalent to establishing the supremum of $e_p(\underline{f})$ on $F'_{L,\varepsilon}$. Since $F'_{L,\varepsilon}$ is contained in $F'_{L,\varepsilon}$, the energy constraint in expression (16) is also implied in $F'_{L,\varepsilon}$. The maximization of the error $e_p(\underline{f})$ on $F'_{L,\varepsilon}$ reduces then to an optimization problem with two inequality constraints; where one of the constraints is given by (16), and the other one is given by the eigenvalue inequality in proposition 2. We state the above optimization problem for $e_p(\underline{f})$ on $F'_{L,\varepsilon}$ and its solution in a theorem.

<u>Theorem 3</u>

The supremum of $e_p(\underline{f})$ *on* $F'_{L,\varepsilon}$ *in (C) corresponds to the solution of the following optimization problem:*

$$\sup \int_{-\pi}^{\pi} \ell og\ [\prod_{j=1}^{n} \lambda_i(\omega)]\ d\omega$$

subject to:

$$\int_{-\pi}^{\pi} \left[\sum_{i=1}^{n} \lambda_i(\omega)\right] d\omega \leq (1-\varepsilon) \int_{-\pi}^{\pi} \left[\sum_{i=1}^{n} \lambda_i^o(\omega)\right] d\omega + \varepsilon\ 2\pi\ W$$

$$\lambda_j(\omega) \geq (1-\varepsilon)\ \lambda_j^o(\omega)\ ;\ \forall\ j,\ \forall\ \omega \in [-\pi,\pi]$$

; where $\{\lambda_j^o(\omega)\ ;\ 1 \leq j \leq n\}$ the eigenvalues of the matrix $\underline{f}_o(\omega)$.

The unique solution of the above optimization problem is represented by an n-tuple $(\lambda_1^*(\omega),\ldots,\lambda_n^*(\omega))$ of eigenvalues such that:

$$\lambda_j^*(\omega) = \begin{cases} (1-\varepsilon)\ \lambda_j^o(\omega) & ;\ \omega : (1-\varepsilon)\ \lambda_j^o(\omega) > \gamma \\ \gamma & ;\ \omega : (1-\varepsilon)\ \lambda_j^o(\omega) \leq \gamma \end{cases} \quad ;\ 1 \leq j \leq n \tag{22}$$

The constant γ is uniquely defined and it is the same for all j values.

Proof

The proof is parallel to the proof of lemma 1. The Kuhn-Tucker conditions here are:

$$\lambda_i^{-1}(\omega) - \gamma^{-1} - \mu_i(\omega) = 0 \quad ; \quad 1 \leq i \leq n$$

$$\mu_i(\omega)\ [\lambda_i(\omega) - (1-\varepsilon)\ \lambda_i^o(\omega)] = 0 \quad ; \quad 1 \leq i \leq n$$

The constant γ and the corresponding regions ($E_{i,\gamma}$, $E_{i,\gamma}^c$; $1 \leq i \leq n$) where each eigenvalue $\lambda_i(\omega)$ is equal to γ or larger than it are determined by the equation:

$$\sum_{i=1}^{n} \left[(1-\varepsilon) \int_{E_{i,\gamma}} \lambda_i^o(\omega)\ d\omega + \gamma \int_{E_{i,\gamma}^c} d\omega \right] =$$

$$= (1-\varepsilon) \sum_{i=1}^{n} \int_{-\pi}^{\pi} \lambda_i^o(\omega)\ d\omega + \varepsilon\ 2\pi\ W \tag{23}$$

Defining Q as in (20) and

$$V_i \triangleq \int_{-\pi}^{\pi} \lambda_i^o(\omega)\ d\omega \tag{24}$$

we can write expression (23) in the following form:

$$\beta(\gamma) \triangleq \sum_{i=1}^{n} \int_{E_{i,\gamma}^c} [\gamma - (1-\varepsilon)\ \lambda_i^o(\omega)]\ d\omega = n\ Q - (1-\varepsilon) \sum_{i=1}^{n} V_i \tag{25}$$

As in the proof of lemma 1, the monotonicity and nonnegativeness of $\beta(\gamma)$ is easily established. It is also true that the right hand part of (25) is positive. Thus, the uniqueness of γ that satisfies the equation in (25).

We exhibit the solution in (22) graphically, in figure 2. We will now summarize our conclusions for the prediction game on the class $F_{L,\varepsilon}$, with some comments.

Comments

Since the error expression $e_p(\underline{f})$ in (14) involves only the eigenvalues of the spectral density matrix $\underline{f}(\omega)$, and since the description of the class $F_{L,\varepsilon}$ in (A) does not involve eigenvectors explicitly, we may adopt several assumptions in terms of the $\underline{f}(\omega)$ eigenvectors. In this section we took two extreme directions. We first assumed that the eigenvectors are unrestricted. Then, we assumed that the eigenvectors remain fixed for the whole class. In the first case, the solution of the prediction game is given by theorem 2. In the second case, the solution is given by theorem 3. The two cases can be looked upon respectively as an upper and a lower bound to the prediction games with any restrictions on the eigenvectors. The case represented by theorem 2 is the most robust within the linear class $F_{L,\varepsilon}$; where any possible restrictions on the eigenvectors are not specified. It is also the most pessimistic. Theorem 3 represents the least robust case within the general linear model of the class in (A). The solution of this case does not safeguard against possible perturbations of the eigenvectors of the matrix $\underline{h}(\omega)$.

We now focus on the interpolation game on $F_{L,\varepsilon} x S_i$. As we pointed out earlier, the search for the solution of this game is more involved. This is so, because the error expression $e_i(\underline{f})$ in (15) involves explicitly both the eigenvalues and the eigenvectors of the spectral density matrix $\underline{f}(\omega)$. For that reason, we will create a lower bound on $e_i(\underline{f})$, that will be strictly a function of the eigenvalues of the matrix $\underline{f}(\omega)$. We will find a supremum of this lower bound on $F_{L,\varepsilon}$. We will search for this supremum by going first through the class F in (D), as we did in the prediction game. This supremum will provide, in general, a lower bound on the saddle value of the interpolation game on $F_{L,\varepsilon} x S_i$. We proceed with the following proposition.

Proposition 3

The error $e_i(\underline{f})$ in (15) is bounded from below by the function

$$b_L(\underline{f}) = 4\,\pi^2\, n^2\left[\int_{-\pi}^{\pi} d\omega \sum_{i=1}^{n} \lambda_i^{-1}(\omega)\right]^{-1} \qquad (26)$$

; where $\{\lambda_i(\omega)\ ;\ 1 \leq i \leq n\}$ the eigenvalues of the spectral density matrix $\underline{f}(\omega)$.

The function $b_L(\underline{f})$ is strictly concave on F and $F_{L,\varepsilon}$ and it is equal to the error $e_i(\underline{f})$ if and only if $\lambda_i(\omega) = \lambda_j(\omega)$; $\forall$ i, j.

Proof

We will use the known identity tr $A^{-1} \geq n^2$ [tr A]$^{-1}$; where A an nxn positive definite nonsingular matrix. The above identity is satisfied with equality iff A is the identity matrix times a constant. Applying the above expression to the error $e_i(\underline{f})$, we find:

$$e_i(\underline{f}) = 4\,\pi^2\ \mathrm{tr}\left[\int_{-\pi}^{\pi} \underline{f}^{-1}(\omega)\, d\omega\right]^{-1} \geq 4\,\pi^2\, n^2\ \mathrm{tr}\left[\int_{-\pi}^{\pi} f^{-1}(\omega)\, d\omega\right]^{-1} =$$

$$= 4\,\pi^2\, n^2\left[\int_{-\pi}^{\pi} d\omega \sum_{i=1}^{n} \lambda_i^{-1}(\omega)\right]^{-1} = b_L(\underline{f})$$

with equality everywhere iff $\int_{-\pi}^{\pi} \underline{f}^{-1}(\omega)\, d\omega$ is the identity matrix times a constant. But this can only occur if all the eigenvalues of $\underline{f}^{-1}(\omega)$ are equal to each other. That, of course, implies that the eigenvalues of $\underline{f}(\omega)$ are equal to each other. A straightforward observation allows us to conclude that $b_L(\underline{f})$ is strictly concave in the set $\{\lambda_i(\omega)\ ;\ 1 \le i \le n\}$; thus, $b_L(\underline{f})$ is strictly concave on F and $F_{L,\varepsilon}$.

We will now seek the supremum of the function $b_L(\underline{f})$ in (26) on the class F in (D). Our approach can be formalized as a constraint optimization problem, as in lemma 1. We present the constraint optimization problem and its solution in a lemma.

Lemma 2

The supremum of the function $b_L(\underline{f})$ in (26) on F in (D) corresponds to the solution of the following constraint optimization problem:

$$\inf\left\{\int_{-\pi}^{\pi} d\omega \sum_{i=1}^{n} \lambda_i^{-1}(\omega)\right\}$$

subject to:

$$\int_{-\pi}^{\pi}\left[\sum_{i=1}^{n} \lambda_i(\omega)\right] d\omega \le (1-\varepsilon)\int_{-\pi}^{\pi}\left[\sum_{i=1}^{n} \lambda_i^{o}(\omega)\right] d\omega + \varepsilon\, 2\pi\, W$$

$$\lambda_j(\omega) \ge (1-\varepsilon)\, \lambda_{max}^{o}(\omega)\ ;\ \forall j\ ,\ \forall\, \omega \in [-\pi,\pi]$$

; where $\lambda_{max}^{o}(\omega)$ is given by (13).

The unique solution of the above optimization problem is identical to the solution (17) in lemma 1.

Proof

A unique solution exists due to the strict concavity of $b_L(\underline{f})$, and the convexity and closeness of F. This unique solution can be found through an Euler-Langrange formalization. The Kuhn-Tucker conditions are:

$$-\lambda_i^{-2}(\omega) + \gamma + \mu_i(\omega) = 0\ ;\ 1 \le i \le n$$

$$\mu_i(\omega)\,[\lambda_i(\omega) - (1-\varepsilon)\, \lambda_{max}^{o}(\omega)] = 0\ ;\ 1 \le i \le n$$

; where $\mu_i(\omega)$; $1 \leq i \leq n$ the Euler-Langrange multipliers.

From the above conditions we obtain:

$$[-\lambda_i^{-2}(\omega) + \gamma]\ [\lambda_i(\omega) - (1-\varepsilon)\ \lambda^{o}_{max}(\omega)] = 0\ ;\ \forall\ i$$

and thus the result in the lemma. The uniqueness of the constant γ is exactly as in lemma 1.

We now proceed with a theorem to exhibit the sufficiency of the solution in lemma 2, for the class $F_{L,\varepsilon}$ as well. The theorem is parallel to theorem 2 in the prediction game.

Theorem 4

The supremum of $b_L(\underline{f})$ on F, given by lemma 2, is also the supremum of $b_L(\underline{f})$ on $F_{L,\varepsilon}$. If $\lambda^*(\omega)$ denotes the solution in lemma 2, then $b_L(\underline{f}^*) \triangleq b_L(\lambda^*) = e_i(\lambda^*) \triangleq e_i(\underline{f}^*)$.

Proof

As in the proof of theorem 2, we select some $\underline{f}$ in $(F_{L,\varepsilon} - F)$. For the eigenvalues $\lambda_1(\omega),\ldots,\lambda_k(\omega)$ of $\underline{f}$ such that they are less than $(1-\varepsilon)\ \lambda^{o}_{max}(\omega)$ for some ω values, we create new eigenvalues $\lambda'_j(\omega)$; $1 \leq j \leq k$, as in the proof of theorem 2. We maintain the remaining eigenvalues unchanged, and we denote the whole newly created set of eigenvalues $\{\lambda'_j(\omega)\ ;\ 1 \leq j \leq n\}$. Then, we have trivially:

$$\int_{-\pi}^{\pi} d\omega \sum_{i=1}^{n} \lambda_i'^{-1}(\omega) \leq \int_{-\pi}^{\pi} d\omega \sum_{i=1}^{n} \lambda_i(\omega)$$

Therefore, for every $\underline{f}$ in $(F_{L,\varepsilon} - F)$, there exists some $\underline{f}'$ in F such that $b_L(\underline{f}') \geq b_L(\underline{f})$. The proof is now complete. $b_L(\underline{f}^*) = e_i(\underline{f}^*)$, because the solution in lemma 2 gives equal eigenvalues and then (due to proposition 3) equality is attained.

We will complete this section with some comments on the interpolation game.

Comments

Due to the involvement of both the eigenvalues and the eigenvectors of the spectral matrix $\underline{f}(\omega)$ in the interpolation error $e_i(\underline{f})$, we were unable to find the general solution of the game in (9) on $F_{L,\varepsilon}xS_i$. Instead, we found a lower bound on the saddle value of this game. The value of this lower bound is given by lemma 2 and theorem 4. It is important to point out, however, that the developed lower bound coincides with the error $e_i(\underline{f})$ for all $\underline{f}$ in F_Q, in at least one case. This is the case where the eigenvectors of all the spectral density matrices in $F_{L,\varepsilon}$ are constant (not functions of ω). This corresponds to the class of spectral density matrices with identical projections on n fixed directions. Then the lower bound $b_L(\underline{f})$ and the error $e_i(\underline{f})$ coincide for all $\underline{f}$.

4. The Solution of the Games for F_Q

In this section we consider the prediction and interpolation games on F_QxS_p and F_QxS_i respectively, where the F_Q class is described by (B) in section 2. Due to theorem 1, the solutions of the above games are provided by the suprema of the errors $e_p(\underline{f})$ and $e_i(\underline{f})$ in (14) and (15) respectively, on the convex and closed class F_Q. We will use the same notation as in section 3, and we will denote by $\{\lambda_i(\omega)\ ;\ 1 \leq i \leq n\}$ the eigenvalues of the spectral density matrix $\underline{f}(\omega)$. We will first analyze the prediction game. Then, we will study the interpolation game. We first define two measures that we will use in our presentation.

Let A be some set in the Borel field B_π. We will denote:

$$\begin{aligned} &m(A)\ :\ \text{The Lebesque measure of A} \\ &1_A(\omega)\ :\ \text{The indicator function of A} \end{aligned} \tag{27}$$

Also, in the description of the class F_Q in (B), we will denote:

$$A_{k+1} = [-\pi,\pi] - \bigcup_{i=1}^{k} A_i$$

$$c_{k+1} = \mathrm{tr} \int_{A_{k+1}} \underline{f}(\omega)\, d\omega = c - \sum_{i=1}^{k} c_i \tag{28}$$

We also use the known identity:

$$\mathrm{tr} \int_A \underline{f}(\omega)\, d\omega = \int_A \left[\sum_{i=1}^{n} \lambda_i(\omega)\right] d\omega \ ; \ A \in B_\pi \tag{29}$$

Due to the identity in (29), and due to the fact that the error $e_p(\underline{f})$ is a function of the eigenvalues of $\underline{f}(\omega)$ only, the maximization of $e_p(\underline{f})$ on F_Q does not involve explicitly the eigenvectors of the spectral density matrices $\underline{f}\ (\omega)$. Indeed, the solution of the prediction game on $F_Q x S_p$ can be formalized as an optimization problem, with constraints of the form as in (29). We express the constraint optimization problem and its solution in a theorem.

Theorem 5

The supremum of $e_p(\underline{f})$ on F_Q in (B) corresponds to the solution of the following optimization problem:

$$\sup \int_{-\pi}^{\pi} \ell og\, [\prod_{i=1}^{n} \lambda_i(\omega)]\, d\omega$$

subject to:

$$\int_{A_j} \left[\sum_{i=1}^{n} \lambda_i(\omega)\right] d\omega = c_j \ ; \ j=1,\ldots,k+1$$

; where $(c_j,\ A_j)$; $1 \leq j \leq k+1$ the pairs given in the description of the class F_Q and in (28).

The unique solution of the above optimization problem is represented by an n-tuple $(\lambda_1^*(\omega),\ldots,\lambda_n^*(\omega))$ of eigenvalues, such that:

$$\forall\ i\ ;\ \lambda_i^*(\omega) = \lambda^*(\omega) = n^{-1} \sum_{j=1}^{k+1} c_j\ 1_{A_j}(\omega)\ m^{-1}(A_j) \tag{30}$$

; where $1_A(\omega)$ and $m(A)$ are given by (27).

Proof

The exitence of a unique solution is guaranteed by theorem 1. It is thus sufficient to prove that:

$$\int_{-\pi}^{\pi} \ell og\ [\lambda^*(\omega)]^n\ d\omega \geq \int_{-\pi}^{\pi} \ell og\ [\prod_{i=1}^{n} \lambda_i(\omega)]\ d\omega\ ;\ \forall\ \{\lambda_i(\omega)\}\ \text{in}\ F_Q \tag{31}$$

; where $\lambda^*(\omega)$ is given by (30).

The eigenvalues $\{\lambda_i(\omega)\}$ are positive. We will use the *inequality between arithmetic and geometric means* for positive numbers, as well as the generalized form of this inequality, known as Jensen's inequality. We thus obtain:

$$\int_{-\pi}^{\pi} \ell og\ [\prod_{i=1}^{n} \lambda_i(\omega)]\ d\omega \leq \int_{-\pi}^{\pi} \ell og \left[n^{-1} \sum_{i=1}^{n} \lambda_i(\omega)\right]^n d\omega = n\int_{-\pi}^{\pi} \ell og \left[n^{-1} \sum_{i=1}^{n} \lambda_i(\omega)\right] d\omega$$

$$= n \sum_{j=1}^{k+1} \int_{A_j} \ell og \left[n^{-1} \sum_{i=1}^{n} \lambda_i(\omega)\right] d\omega \leq n \sum_{j=1}^{k+1} m(A_j)\ \ell og \left\{m^{-1}(A_j) \int_{A_j} n^{-1}\left[\sum_{i=1}^{n} \lambda_i(\omega)\right] d\omega\right\}$$

$$= n \sum_{j=1}^{k+1} m(A_j)\ \ell og\left\{n^{-1}\ m^{-1}(A_j)\ c_j\right\} = n \sum_{j=1}^{k+1} \int_{A_j} \ell og \left\{n^{-1}\ m^{-1}(A_j)\ c_j\right\} d\omega =$$

$$= n\int_{-\pi}^{\pi} \ell og \left\{\sum_{j=1}^{k+1} n^{-1}\ m^{-1}(A_j)\ c_j\ 1_{A_j}(\omega)\right\} d\omega = \int_{-\pi}^{\pi} \ell og[\lambda^*(\omega)]^n\ d\omega \tag{32}$$

The sequence of equalities and inequalities in (32) hold with strict equality everywhere, if and only if $\lambda_i(\omega) = \lambda^*(\omega)$; $\forall$ i. The proof is now complete.

We exhibit the solution in (30) graphically, in figure 3. We will complete the coverage of the prediction game in the class $\mathcal{F}_Q$ with some somments.

Comments

The solution in theorem 5 implies no restrictions on the eigenvectors of the spectral density matrices $\underline{f}(\omega)$ in the class $\mathcal{F}_Q$. Indeed, the class $\mathcal{F}_Q$ imposes quantiled energy restrictions only. It leaves the projections of the spectral density matrices unrestricted. The solution in (30) gives spectra that are piecewise constant functions in $[-\pi,\pi]$. This satisfies our intuition, since we expect the "flatest" possible spectra, as the solution to the prediction game, for any convex and closed class of measures. Indeed, the prediction error has entropy characteristics.

We now consider the interpolation game on $\mathcal{F}_Q x S_i$. As concluded from theorem 1, it is equivalent to search for the supremum of the error $e_i(\underline{f})$ in (15) on the class $\mathcal{F}_Q$. Since the error $e_i(\underline{f})$ involves explicitly both the eigenvalues and the eigenvectors of the spectral density matrix $\underline{f}(\omega)$, we will analyze the lower bound $b_L(\underline{f})$ in (26) instead (as we did in section 3). As in section 3, the function $b_L(\underline{f})$ is strictly concave on $\mathcal{F}_Q$. We will thus search for the unique supremum of $b_L(\underline{f})$ on $\mathcal{F}_Q$. This search can be formalized as a constraint optimization problem. We state the optimization problem and its solution in a lemma.

Lemma 3

The supremum of the function $b_L(\underline{f})$ in (26) on $\mathcal{F}_Q$ corresponds to the solution of the following constraint optimization problem:

$$\inf\left\{\int_{-\pi}^{\pi}\left[\sum_{i=1}^{n}\lambda_i^{-1}(\omega)\right]d\omega\right\}$$

Subject to:

$$\int_{A_j}\left[\sum_{i=1}^{n}\lambda_i(\omega)\right]d\omega = c_j \quad ; \; j=1,\ldots,k+1$$

; where (c_j, A_j) as in theorem 5.

The unique solution of the above optimization problem is identical to the solution (30) in theorem 5. Furthermore, at this solution the values of $b_L(\underline{f})$ and $e_i(\underline{f})$ coincide.

Proof

We follow similar approach as in the proof of theorem 5. Again, since the uniqueness of the solution is established in advance, it is sufficient to show that:

$$\int_{-\pi}^{\pi} n[\lambda^*(\omega)]^{-1}\, d\omega \le \int_{-\pi}^{\pi}\left[\sum_{i=1}^{n}\lambda_i^{-1}(\omega)\right]d\omega \; ; \; \forall\{\lambda_i(\omega)\} \text{ in } F_Q \tag{33}$$

; where $\lambda^*(\omega)$ the solution in theorem 5.

We will use the following inequalities on positive numbers and functions:

$$\sum_{i=1}^{N} a_i^{-1} \ge N^2\left[\sum_{i=1}^{N} a_i\right]^{-1} \tag{34}$$

$$\int_A f^{-1}(x)dx \ge \left[\int_A f(x)dx\right]^{-1} m^2(A) \tag{35}$$

The inequality in (34) represents a relationship between the arithmetic and the harmonic mean of positive numbers. It is satisfied with equality iff $a_i = a$; $\forall$ i. The inequality in (35) in Jensen's inequality, where m(A) is the Lebesque measure of the set A. Expression (35) is satisfied with equality iff f(x) is constant on A. Using the inequalities in (34) and (35) we obtain:

$$\int_{-\pi}^{\pi}\left[\sum_{i=1}^{n}\lambda_i^{-1}(\omega)\right]d\omega=\sum_{j=1}^{k+1}\int_{A_j}\left[\sum_{i=1}^{n}\lambda_i^{-1}(\omega)\right]d\omega\geq\sum_{j=1}^{k+1}n^2\int_{A_j}\left[\sum_{i=1}^{n}\lambda_i(\omega)\right]^{-1}d\omega\geq$$

$$n^2\sum_{j=1}^{k+1}m^2(A_j)\left\{\int_{A_j}\left[\sum_{i=1}^{n}\lambda_i(\omega)\right]d\omega\right\}^{-1}=n^2\sum_{j=1}^{k+1}c_j^{-1}\,m^2(A_j)\qquad(36)$$

But for the solution $\lambda^*(\omega)$ in theorem 5, we obtain:

$$\int_{-\pi}^{\pi}n[\lambda^*(\omega)]^{-1}\,d\omega=n\int_{-\pi}^{\pi}n^{-1}\left[\sum_{j=1}^{k+1}c_j\,m^{-1}(A_j)\,1_{A_j}(\omega)\right]^{-1}d\omega=$$

$$=n^2\int_{-\pi}^{\pi}\left[\sum_{j=1}^{k+1}c_j\,m^{-1}(A_j)\,1_{A_j}(\omega)\right]^{-1}d\omega=n^2\sum_{j=1}^{k+1}\int_{A_j}\left[\sum_{i=1}^{k+1}c_i\,m^{-1}(A_i)\,1_{A_i}(\omega)\right]^{-1}d\omega$$

$$=n^2\sum_{j=1}^{k+1}c_j^{-1}\,m^2(A_j)\qquad(37)$$

From (36) and (37) we conclude directly that (33) is satisfied. Furthermore, the inequalities in (36) are strict equalities iff $\lambda_i(\omega)=\lambda^*(\omega)$; $\forall$ i. Thus, the left part of (33) is attained. Finally, as in the proof of theorem 4, if $\underline{f}^*$ is some spectral density matrix with eigenvalues $\lambda_i(\omega)=\lambda^*(\omega)$; $\forall$ i, we also have $e_i(\underline{f}^*)=b_L(\underline{f}^*)$. The proof is now complete.

We will conclude this section with some comments on the interpolation game on $F_Q xS_i$.

Comments

As in section 3, the involvement of eigenvectors in the error expression $e_i(\underline{f})$ prevented us from finding a general solution of the interpolation game on $F_Q x S_i$. Instead, we maximized a lower bound on the saddle value of the game. The supremum of this lower bound $b_L(\underline{f})$ is equal to the error $e_i(\underline{f})$, at the $\underline{f}$ value that satisfies the supremum $\sup_{\underline{f} \in F_Q} b_L(\underline{f})$. If the spectral density matrices in F_Q have identical constant eigenvectors, the lower bound $b_L(\underline{f})$ coincides with the error $e_i(\underline{f})$ for all $\underline{f}$. Then, the solution in lemma 3 is also the solution of the interpolation game.

5. Conclusions

In this paper, we considered the prediction and interpolation problems for vector stationary processes with ill-specified statistical structures. We modeled the uncertainty in the statistical description of the processes through convex and closed families of vector measures. We formalized the prediction and interpolation problems as statistical games with saddle point solutions, and we considered two different families of vector measures. The first such family represents a linear contamination of a nominal measure, and it includes an energy constraint. We provided two solutions of the prediction game within this family. One of the solutions is the most robust, while the second solution is the least robust. They are both represented by specific choices of the eigenvalues of spectral density matrices. Within the same family, we analyzed a lower bound on the solution of the interpolation game, and we found the conditions under which this lower bound is attained.

The second family of vector measures we considered is represented by fixed energy, on a finite number of prespecified frequency quantiles. The solution of the prediction game is then provided by identical eigenvalues such that each is piece-wise constant. For the interpolation game, we analyzed a lower bound instead, and we found again the conditions under which this lower bound is attained.

All the derived solutions for the prediction game correspond to the eigenvalues with the "flatest" possible tails. Equivalently, these solutions correspond to the measures with the most evenly spread energy. This is intuitively satisfactory, since the prediction error has entropy characteristics in the frequency domain, and the most even spreading results in entropy maximization.

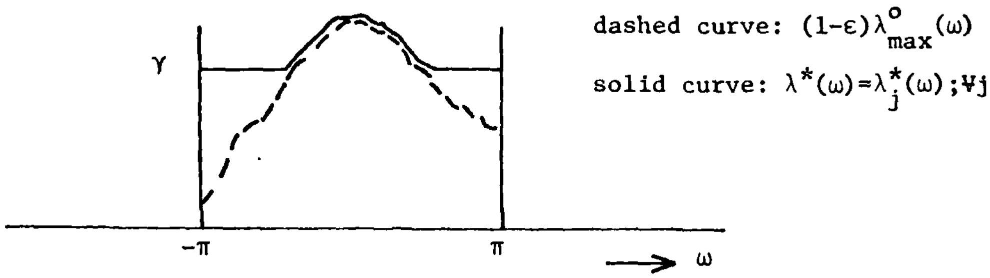

Figure 1

The Solution on $F_{L,\varepsilon}$

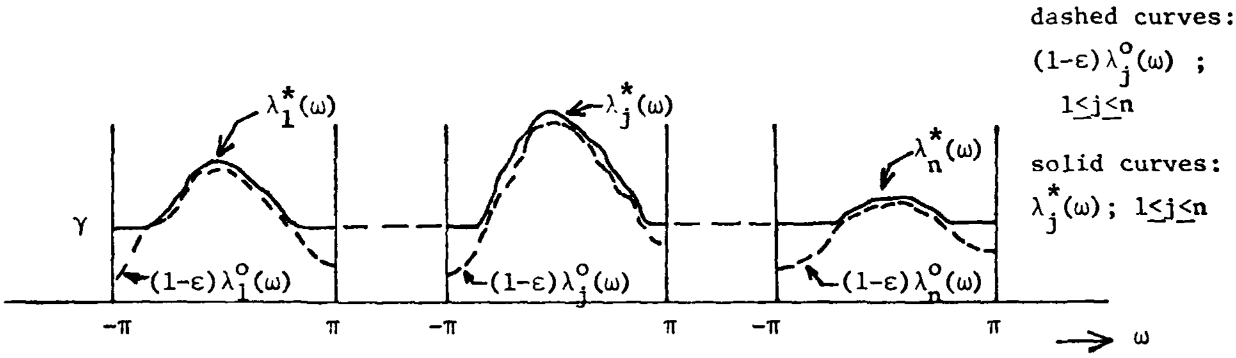

Figure 2

The Solution on $F'_{L,\varepsilon}$

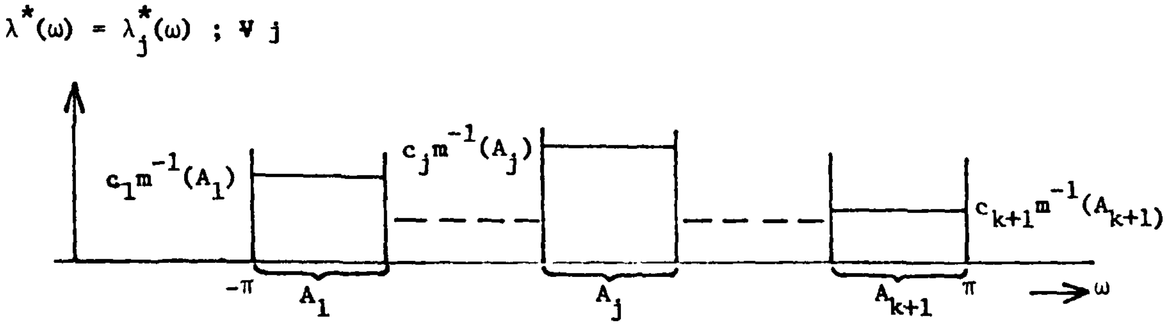

Figure 3

The Solution on F_Q

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7. Appendix

Proof of Theorem 1

We will prove the theorem through a sequence of lemmas.

Lemma A.1

Let A be an nxn symmetric and positive definite matrix. Then,

i) $f(A) \overset{\Delta}{=} \operatorname{tr} \ell og\, A$

is strictly concave in A

ii) $h(A) \overset{\Delta}{=} \operatorname{tr} A^{-1}$

is strictly convex in A

Proof

Let M be the set of all nxn symmetric and positive definite matrices. M is convex and closed. Thus, if $A_1 \in M$, $A_2 \in M$, then $[\alpha A_1 + (1-\alpha) A_2] \in M$; $\forall \alpha$: $0 \leq \alpha \leq 1$.

To prove the lemma, it suffices to show that:

$$\frac{\partial^2}{\partial\alpha^2} f\, (\alpha A_1 + (1-\alpha) A_2) < 0 \; ; \forall A_1, A_2 \in M, A_1 \neq A_2$$

$$\frac{\partial^2}{\partial\alpha^2} h\, (\alpha A_1 + (1-\alpha) A_2) > 0 \; ; \forall A_1, A_2 \in M, A_1 \neq A_2$$

But, we easily obtain:

i)
$$\frac{\partial^2}{\partial\alpha^2} f\, (\alpha A_1 + (1-\alpha) A_2) = \operatorname{tr} \frac{\partial^2}{\partial\alpha^2} \ell og\, [\alpha A_1 + (1-\alpha) A_2] =$$

$$= \operatorname{tr} \frac{\partial}{\partial\alpha} \Big[[\alpha A_1 + (1-\alpha) A_2]^{-1} [A_1 - A_2] \Big]$$

$$= - \operatorname{tr} [\alpha A_1 + (1-\alpha) A_2]^{-2} [A_1 - A_2] [A_1 - A_2] =$$

$$= - \operatorname{tr} [A_1 - A_2]^T [\alpha A_1 + (1-\alpha) A_2]^{-2} [A_1 - A_2] = - \sum_{i=1}^{n} c_i^T [\alpha A_1 + (1-\alpha) A_2]^{-2} c_i \tag{A.1}$$

; where c_i the ith column of the symmetric matrix $A_1 - A_2$.

Since $\alpha A_1 + (1-\alpha) A_2$ is positive definite, so will be the matrix $[\alpha A_1 + (1-\alpha) A_2]^{-2}$. Also, if $A_1 \neq A_2$ some columns c_i will be nonzero. Thus, the expression in (A.1) is strictly negative.

ii) $$\frac{\partial^2}{\partial\alpha^2} h(\alpha A_1 + (1-\alpha) A_2) = \operatorname{tr} \frac{\partial^2}{\partial\alpha^2} [\alpha A_1 + (1-\alpha) A_2]^{-1} =$$

$$= -\operatorname{tr} \frac{\partial}{\partial\alpha} [\alpha A_1 + (1-\alpha) A_2]^{-2} [A_1 - A_2]$$

$$= 2 \operatorname{tr} [\alpha A_1 + (1-\alpha) A_2]^{-3} [A_1 - A_2] [A_1 - A_2] =$$

$$= 2 \operatorname{tr} [A_1 - A_2]^T [\alpha A_1 + (1-\alpha) A_2]^{-3} [A_1 - A_2] \qquad \text{(A.2)}$$

If $\alpha A_1 + (1-\alpha) A_2$ is a positive definite matrix, so is $[\alpha A_1 + (1-\alpha) A_2]^{-3}$. As in i), for $A_1 \neq A_2$ the expression in (A.2) is strictly positive.

Lemma A.2

The errors $e_p(\mu)$ and $e_i(\mu)$ are strictly concave in F. Thus, unique μ_p^* and μ_i^* exist that satisfy the suprema in (10) and (11).

Proof

i) The strict concavity of $e_p(\mu)$ is directly due to part i) of lemma A.1 and the fact that $\mu_1 \in F$, $\mu_2 \in F \rightarrow [\varepsilon \mu_1 + (1-\varepsilon) \mu_2] \in F$; $\forall \varepsilon: 0 \leq \varepsilon \leq 1$, and $\underline{f}_{\varepsilon\mu_1+(1-\varepsilon)\mu_2}(\omega) = \varepsilon \underline{f}_{\mu_1}(\omega) + (1-\varepsilon) \underline{f}_{\mu_2}(\omega)$.

ii) To prove the strict concavity of $e_i(\mu)$, we select $\mu_1, \mu_2 \in F$, and $\varepsilon : 0 \leq \varepsilon \leq 1$. Then, $[\varepsilon \mu_1 + (1-\varepsilon) \mu_2] \in F$, and we can write from (7):

$$[4\pi^2]^{-1} e_i(\varepsilon \mu_1 + (1-\varepsilon) \mu_2) = \operatorname{tr}\left[\int_{-\pi}^{\pi} [\varepsilon \underline{f}_{\mu_1}(\omega) + (1-\varepsilon) \underline{f}_{\mu_2}(\omega)]^{-1} d\omega\right]^{-1} \qquad \text{(A.3)}$$

Using the identity $\frac{\partial}{\partial\alpha} A^{-1}(\alpha) = -A^{-1}(\alpha) \left[\frac{\partial}{\partial\alpha} A(\alpha)\right] A^{-1}(\alpha)$ for matrices and differentiating expression (A.3) twice with respect to ε, we find:

$$\frac{\partial^2}{\partial\varepsilon^2} [4\pi^2]^{-1} e_i(\varepsilon\ \mu_1 + (1-\varepsilon)\ \mu_2) =$$

$$= -2\mathrm{tr} \left\{ A^{-1}\left(\int_{-\pi}^{\pi} B^{-1} \dot{B} B^{-1} \dot{B} B^{-1} + \int_{-\pi}^{\pi} B^{-1} \dot{B} B^{-1}\left[\int_{-\pi}^{\pi} B^{-1}\right]^{-1} \int_{-\pi}^{\pi} B^{-1} \dot{B} B^{-1}\right) A^{-1} \right\} \tag{A.4}$$

; where:

$$B = \varepsilon\ f_{\mu_1}(\omega) + (1-\varepsilon)\ f_{\mu_2}(\omega)$$

$$A = \int_{-\pi}^{\pi} [\varepsilon\ f_{\mu_1}(\omega) + (1-\varepsilon)\ f_{\mu_2}(\omega)]^{-1}\, d\omega$$

$$\dot{B} = \frac{\partial}{\partial\varepsilon} B$$

The terms in (A.4) are positive definite, thus the total expression is negative.

Lemma A.3

The measures μ_p^* and μ_i^* of lemma A.2 and the corresponding predictor g_p^* and interpolator g_i^* in theorem 1 satisfy respectively the games in (8) and (9).

Proof

Let us consider the error expressions $e_p(\mu, \underline{g}_p)$ and $e_i(\mu, \underline{g}_i)$ given by (4) and (5) respectively. Both $e_p(\mu, \underline{g}_p)$ and $e_i(\mu, \underline{g}_i)$ are linear in μ. Also, since F is a closed family, the suprema $\sup_{\mu\varepsilon F} e_p(\mu, \underline{g}_p)$, and $\sup_{\mu\varepsilon F} e_i(\mu, \underline{g}_i)$ exist and are attained in F. For given $\underline{g}_p$ in S_p and given $\underline{g}_i$ in S_i, let:

$$\sup_{\mu\varepsilon F} e_p(\mu, \underline{g}_p) = e_p(\mu_{\underline{g}_p}, \underline{g}_p)$$

$$\sup_{\mu\varepsilon F} e_i(\mu, \underline{g}_i) = e_i(\mu_{\underline{g}_i}, \underline{g}_i)$$

Then, due to lemma A.2, we obtain:

$$\inf_{\underline{g}_p \varepsilon S_p} \sup_{\mu \varepsilon F} e_p(\mu,\underline{g}_p) = \inf_{\underline{g}_p \varepsilon S_p} e_p(\mu_{\underline{g}_p},\underline{g}_p) \leq \inf_{\underline{g}_p^o \varepsilon S_p} e_p(\mu_{\underline{g}_p},\underline{g}_p^o) = e_p(\mu_{\underline{g}_p}) \leq e_p(\mu_p^*)$$

$$\inf_{\underline{g}_i \varepsilon S_i} \sup_{\mu \varepsilon F} e_i(\mu,\underline{g}_i) = \inf_{\underline{g}_i \varepsilon S_i} e_i(\mu_{\underline{g}_i},\underline{g}_i) \leq \inf_{\underline{g}_i^o \varepsilon S_i} e_i(\mu_{\underline{g}_i},\underline{g}_i^o) = e_i(\mu_{\underline{g}_i}) \leq e_i(\mu_i^*) \tag{A.5}$$

But by definition:

$$e_p(\mu_p^*) = \sup_{\mu \varepsilon F} \inf_{\underline{g}_p \varepsilon S_p} e_p(\mu,\underline{g}_p)$$

$$e_i(\mu_i^*) = \sup_{\mu \varepsilon F} \inf_{\underline{g}_i \varepsilon S_i} e_i(\mu,\underline{g}_i) \tag{A.6}$$

Thus, from (A.5) and (A.6) we obtain:

$$\inf_{\underline{g}_p \varepsilon S_p} \sup_{\mu \varepsilon F} e_p(\mu,\underline{g}_p) \leq \sup_{\mu \varepsilon F} \inf_{\underline{g}_p \varepsilon S_p} e_p(\mu,\underline{g}_p)$$

$$\inf_{\underline{g}_i \varepsilon S_i} \sup_{\mu \varepsilon F} e_i(\mu,\underline{g}_i) \leq \sup_{\mu \varepsilon F} \inf_{\underline{g}_i \varepsilon S_i} e_i(\mu,\underline{g}_i) \tag{A.7}$$

But it is always true that $\sup_x \inf_y f(x,y) \leq \inf_y \sup_x f(x,y)$ for any function $f(x,y)$. Due to that, expressions (A.7) must hold with equality. Thus, the games have unique solutions given by the statement of the lemma.

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